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Restricted Constant of Motion for the One-Dimensional Harmonic Oscillator With Quadratic Dissipation and Some Consequences in Statistic and Quantum Mechanics

G. López[1]

A restricted constant of motion, Lagrangian and Hamiltonian, for the harmonic oscillator with quadratic dissipation is deduced. The restriction comes from the consideration of the constant of motion for the velocity of the particle either for $v \geq 0$ or for $v < 0$. A study is done about the implications that these restricted variables have on the specific heat of a thermodynamical system of oscillators with this dissipation, and on the quantization of this dissipative system.

1. INTRODUCTION

There have been many attempts to quantize dissipative system (Cantrijn, 1982; Potasek and Yunke, 1980; Ran and Griffin, 1974; Senitzky, 1960) through Hamiltonian formalism. However, the consistency and the real meaning of these Hamiltonians is quite questionable. For one-dimensional systems, it is known that one way to obtain a consistent meaningful Hamiltonian for a dissipative system is to first get the constant of motion associated with this system (López, 1996a). Once a correct Hamiltonian for a dissipative system is obtained, it is meaningful to study its possible applications in statistic mechanics (López, 1996b) or quantum mechanics (López and Sosa, to appear).

In this paper, the constant of motion approach is used to find a Hamiltonian for one-dimensional dissipative harmonic oscillator, where the dissipation depends quadratically on the velocity. In addition, this Hamiltonian is used to study statistic properties as well as the quantized characteristics of this dissipative system.

[1] Departamento de Física, de la Universidad de Guadalajara, Apartado Postal 4-137, 44410 Guadalajara, Jalisco, México.

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2. RESTRICTED CONSTANT OF MOTION

The constant of motion $K_\alpha = K_\alpha(x, v)$ of the one-dimensional harmonic oscillator with quadratic dependence in the velocity

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = -\omega^2 x - \alpha v|v| \tag{1}$$

satisfies the equation

$$v\frac{\partial K_\alpha}{\partial x} - \left(\omega^2 x + \frac{\alpha v}{m}|v|\right)\frac{\partial K_\alpha}{\partial v} = 0, \tag{2}$$

where α is the parameter which characterizes the dissipation, ω is the natural frequency of oscillations, and $|v|$ is the absolute value of the velocity

$$|v| = \begin{cases} v, & \text{if } v > 0 \\ -v, & \text{if } v < 0. \end{cases} \tag{3}$$

The solution of Eq. (2) is given by

$$K_\alpha = \begin{cases} \frac{1}{2}mv^2 \exp(2\alpha x/m) + \frac{m\omega^2}{(2\alpha/m)^2}[(2\alpha x/m - 1)\exp(2\alpha x/m) + 1], & \text{if } v > 0 \\ \frac{1}{2}mv^2 \exp(-2\alpha x/m) + \frac{m\omega^2}{(2\alpha/m)^2}[1 - (2\alpha x/m + 1)\exp(-2\alpha x/m)], & \text{if } v < 0. \end{cases} \tag{4}$$

One may call this expression as “restricted constant of motion” (r-constant of motion, for short) since its value remains constant as long as as the motion of the particle remains with $v \geq 0$ or $v < 0$. Whenever there be a change in the sign of the velocity, the value of (4) changes, as it is shown in the next section.

3. SPIRAL MOTION IN (x, v) SPACE

Since the r-constant of motion changes for $v < 0$ and $v \geq 0$, it is necessary to take into account this fact to study the motion in the x–v plane. If the initial conditions of Eq. (1) are such that $x(0) = x_0^{(0)} = x_0$ and $v(0) = v_0^{(0)} = v_0 > 0$, the nth trace in the upper plane ($v > 0$) is given (using Eq. (4)) by

$$v = \sqrt{\left[\frac{K_+^{(2n)}}{m} - \frac{2\omega^2}{(2\alpha/m)^2}\right]\exp(-2\alpha x/m) - \frac{2\omega^2}{(2\alpha/m)^2}\left[\frac{2\alpha}{m}x - 1\right]}, \tag{5a}$$

where $K_+^{(2n)}$ is defined as

$$K_+^{(2n)} = \frac{m\omega^2}{(2\alpha/m)^2}\left[\frac{2\alpha}{m}x_0^{(2n)} - 1\right]\exp\left(2\alpha x_0^{(2n)}/m\right) + \frac{m\omega^2}{(2\alpha/m)^2} \tag{5b}$$

and $K_0^{(0)}$ is given by

$$K_0^{(0)} = \frac{1}{2}m\exp(2\alpha x_0/m)\left[v_0^2 + \frac{2\omega^2}{(2\alpha/m)^2}(2\alpha x_0/m - 1)\right] + \frac{m\omega^2}{(2\alpha/m)^2}. \tag{5c}$$

The quantity $x_0^{(2n)}$ is calculated from the equation

$$K_-^{(2n-1)} = -\frac{m\omega^2}{(2\alpha/m)^2}\left[\frac{2\alpha}{m}x_0^{(2n)} + 1\right]\exp\left(2\alpha x_0^{(2n)}/m\right) + \frac{m\omega^2}{(2\alpha/m)^2}, \tag{5d}$$

where the constant $K_-^{(2n-1)}$ is the constant defined previously for the lower plane $v < 0$. The condition $v = 0$ in (5a) defines the point $x_0^{(2n+1)}$; therefore

$$K_+^{(2n)} = \frac{m\omega^2}{(2\alpha/m)^2}\left[\frac{2\alpha}{m}x_0^{(2n+1)} - 1\right]\exp\left(2\alpha x_0^{(2n+1)}/m\right) + \frac{m\omega^2}{(2\alpha/m)^2} \tag{5e}$$

which determine the condition $(x_0^{(2n+1)}, 0)$ for the definition of the r-constant of motion $K_-^{(2n+1)}$ for the lower plane $v < 0$,

$$K_-^{(2n+1)} = -\frac{m\omega^2}{(2\alpha/m)^2}\left[\frac{2\alpha}{m}x_0^{(2n+1)} - 1\right]\exp\left(2\alpha x_0^{(2n+1)}/m\right) + \frac{m\omega^2}{(2\alpha/m)^2}. \tag{6a}$$

Given this r-constant of motion, the trace in the x–v plane for $v < 0$ can be done through Eq. (4),

$$v = -\sqrt{\left[\frac{2K_-^{(2n+1)}}{m} - \frac{2\omega^2}{(2\alpha/m)^2}\right]\exp(2\alpha x/m) + \frac{2\omega^2}{(2\alpha/m)^2}\left[\frac{2\alpha}{m}x + 1\right]}, \tag{6b}$$

and the point $(x_0^{(2n+2)}, 0)$, defined by setting $v = 0$ in Eq. (6b).

$$K_-^{(2n+1)} = -\frac{m\omega^2}{(2\alpha/m)^2}\left[\frac{2\alpha}{m}x_0^{(2n+2)} + 1\right]\exp\left(2\alpha x_0^{(2n+2)}/m\right) + \frac{m\omega^2}{(2\alpha/m)^2} \tag{6c}$$

establishes the condition to determine the constant $K_+^{(2n+2)}$ for $v > 0$, and so on. This procedure brings about the expected shrinking spiral dynamical behavior on the x–v space. If the initial condition is such that $v_0 < 0$, then one starts with the constant for $v < 0$ and follows the same procedure.

4. RESTRICTED LAGRANGIAN AND HAMILTONIAN

Using the solution of "the one-dimensional inverse problem of the mechanics" (Darboux, 1894) to find the restricted Lagrangian (r-Lagrangian) in terms of the r-constant of motion (Kobusen, 1979; López and Hernández, 1989),

$$L(x, v) = v \int^{v} \frac{K_\alpha(x, \xi)}{\xi^2}\, d\xi, \tag{7}$$

the r-Lagrangian associated with the system (1) is

$$L_\alpha = \begin{cases} \frac{1}{2} m v^2 \exp(2\alpha x/m) - \frac{m\omega^2}{(2\alpha/m)^2}[(2\alpha x/m - 1)\exp(2\alpha x/m) + 1], & \text{if } v \geq 0 \\ \frac{1}{2} m v^2 \exp(-2\alpha x/m) - \frac{m\omega^2}{(2\alpha/m)^2}[1 - (2\alpha x/m + 1)\exp(-2\alpha x/m)], & \text{if } v < 0, \end{cases} \tag{8}$$

and the restricted generalized (r-generalized) linear momentum is

$$p_\alpha = \begin{cases} mv \exp(2\alpha x/m), & \text{if } v \geq 0 \\ mv \exp(-2\alpha x/m), & \text{if } v < 0 \end{cases} + [L]_0(x)\,\delta(v), \tag{9a}$$

where $\delta(v)$ is the distribution delta of Dirac (Gel'fand and Shilov, 1968) and $[L]_0(x)$ is the discontinuity jump of (8) at $v = 0$,

$$[L]_0(x) = \frac{m^3\omega^3}{2\alpha^2}\left\{\sinh(2\alpha x/m) - \frac{2\alpha x}{m}\cosh(2\alpha x/m)\right\}. \tag{9b}$$

From (9a) knowing v as a function of p_α and x is complicated because of the generalized function $\delta(v)$. However, one notices that if (9a) is used in the Legendre transformation $H = pv - L$, the resulting term $[L]_0(x)v\delta(v)$ is canceled ($v\delta(v) = 0$). So, the appearing of a δ term in the Hamiltonian is a slightly subtle matter. Moreover, one could study the case where things happen "almost everywhere" (Hewitt and Stromberg, 1965). In this case, v as a function of p_α and x is given by

$$mv = \begin{cases} p_\alpha \exp(-2\alpha x/m), & \text{if } p_\alpha \geq 0 \\ p_\alpha \exp(2\alpha x/m), & \text{if } p_\alpha < 0. \end{cases} \tag{10}$$

Substituting v from Eq. (10) into Eq. (4), one gets the restrictive Hamiltonian (r-Hamiltonian)

$$H_\alpha = \begin{cases} \frac{p_\alpha}{2m} \exp(-2\alpha q/m) + \frac{m\omega^2}{(2\alpha/m)^2}[(2\alpha q/m - 1)\exp(2\alpha q/m) + 1], & \text{if } p_\alpha \geq 0 \\ \frac{p_\alpha}{2m} \exp(2\alpha q/m) + \frac{m\omega^2}{(2\alpha/m)^2}[1 - (2\alpha q/m + 1)\exp(-2\alpha q/m)], & \text{if } p_\alpha < 0. \end{cases} \tag{11}$$

The Hamiltonian's equations of motion for $p_\alpha > 0$ are given by

$$\begin{aligned} \dot{q} &= \frac{p_\alpha}{m}\, \exp(-2\alpha q/m) \\ \dot{p}_\alpha &= \frac{\alpha p_\alpha^2}{m^2}\, \exp(-2\alpha q/m) - m\omega^2 q\, \exp(2\alpha q/m), \end{aligned} \tag{12a}$$

and for $p_\alpha < 0$ are given by

$$\begin{aligned} \dot{q} &= \frac{p_\alpha}{m}\, \exp(2\alpha q/m) \\ \dot{p}_\alpha &= -\frac{\alpha p_\alpha^2}{m^2}\, \exp(2\alpha q/m) - m\omega^2 q\, \exp(-2\alpha q/m). \end{aligned} \tag{12b}$$

The difference between the dynamical system (1) and (12) is evident. Equations (4), (8), (9a), (10), and (11) have the right limit for $\alpha \to 0$. For dissipation at first order in the parameter α, the r-constant of motion and the r-Hamiltonian are given by

$$K = \begin{cases} \dfrac{m}{2}(v^2 + \omega^2 x^2) + \alpha(\omega^2 x^3 + v^2 x), & \text{if } v \geq 0 \\ \dfrac{m}{2}(v^2 + \omega^2 x^2) - \alpha(\omega^2 x^3 + v^2 x), & \text{if } v < 0 \end{cases} \tag{13a}$$

and

$$H = \begin{cases} p^2/2m + \dfrac{1}{2}m\omega^2 q^2 + \alpha(\omega^2 q^3 - p^2 q/m), & \text{if } p \geq 0 \\ p^2/2m + \dfrac{1}{2}m\omega^2 q^2 - \alpha(\omega^2 q^3 - p^2 q/m), & \text{if } p < 0, \end{cases} \tag{13b}$$

where p has the usual expression for the generalized linear momentum ($p = mv$).

5. APPLICATION IN CLASSICAL STATISTIC MECHANICS

Assume a crystal embedded in a boson medium such that the interaction of the atoms with this medium brings about damping in the vibrational degree of freedom of the atoms. Assume also that the damping could be described by Eq. (1) for each atom, where the correlations with the boson system and with other atoms is neglected. Therefore, the overall effect on the vibrational state of the atom is taken into account with the term $-\alpha v|v|$ from Eq. (1). Ignoring the contribution of the boson to the specific heat of the thermodynamical system, one may want to know the effect of dissipation on the specific heat of the vibrational motion of the atoms in the crystal. This effect is now possible to estimate since the Hamiltonian is known (Eq. (8)). To be able to make the approximation using the classical statistic mechanics, the integration appearing in the canonical Gibbs' ensemble

(Huang, 1987) of distinguishable particles,

$$Z = \frac{1}{h^{3N}} \int \exp(-\beta \mathbf{H})\, d\vec{p}\, d\vec{q}, \tag{14a}$$

must be thought in the sense of Lebesgue integration [10], and the function $\exp(-\beta H)$ must be thought in the sense of a "measurable function"[10]. Of course, the following assumptions are followed: the Gibbs' ensemble is almost everywhere time independent; ergodicity is stablished on this system for $p > 0$ and $p < 0$; the number of crossing points for going from $p > 0$ to $p < 0$ (or vice versa) forms a numerable set (therefore having measure zero in the phase space). In the expression (14a), N is the number of particles, h is the Planck's constant and $\vec{p}$ and $\vec{q}$ are vectors in $3N$-dimensional phase space representing the momentum and position of each particle, and β is given in terms of the temperature T and the Boltzmann's constant k as $\beta = 1/kT$. The specific heat (at constant volume) is calculated from the equation

$$C_V = \frac{\partial}{\partial T}\left(kT^2 \frac{\partial \log Z}{\partial T}\right). \tag{14b}$$

Since one is assuming no correlation between particles and decoupling of their coordinates, Eq. (14a) can be written as

$$Z = z^{3N}, \tag{15a}$$

where z is the partition function of just one degree of freedom,

$$z = \frac{1}{h} \int \exp(-\beta H)\, dp\, dq = \frac{1}{h} \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} \exp(-\beta H)\, dp. \tag{15b}$$

Using Eq. (11) and doing the integration with respect to the variable p, Eq. (15b) is written as

$$z = \frac{\sqrt{2m\pi kT}}{2h}\left\{ \int_{-\infty}^{+\infty} \exp(-\beta f(q) + \alpha q/m)\, dq + \int_{-\infty}^{+\infty} \exp(-\beta f(-q) - \alpha q/m)\, dq \right\}, \tag{16a}$$

where $f(q)$ is defined as

$$f(q) = \frac{m\omega^2}{(2\alpha/m)^2}\left[1 + \left(\frac{2\alpha}{m} q - 1\right) \exp(2\alpha q/m)\right]. \tag{16b}$$

This function is written at first order of approximation in α as

$$f(q) \approx \frac{1}{2} m\omega^2 q^2 + \frac{2}{3}\alpha\omega^2 q^3. \tag{16c}$$

Using this approximation in (16a) and a Taylor expansion of the term $\exp(-\beta 2\alpha\omega^2 q^3/3)$, one gets

$$z = \frac{kT}{2\hbar w}\ \exp(\alpha^2 kT/2m^3\omega^2)\left\{1 - \frac{16}{9}\frac{\alpha^2 kT}{m^3\omega^2} + O(\alpha^3)\right\}, \tag{17}$$

where $O(\alpha^3)$ represents the rest of the terms which are of the order higher or equal to α^3. Therefore, at the minimum order in α, one has

$$\log Z = 3N\ \log\left(\frac{kT}{2\hbar\omega}\right) + \frac{\alpha^2 kT}{2m^3\omega^2} + \log\left[1 - \frac{16}{9}\frac{\alpha^2 kT}{m^3\omega^2}\right]. \tag{18}$$

Now, using (14b), the specific heat is given by

$$C_V = 3Nk + \alpha^2\left[\frac{3Nk^2}{2m^3\omega^2} - \frac{16}{3}\frac{Nk^2/m^3\omega^2}{1 - \frac{16}{9}\frac{\alpha^2 kT}{m^3\omega^2}}\right]. \tag{19}$$

The first term in (19) is the classical value for the specific heat, and the second term is the contribution due to dissipation. The singularity appearing in (19) has no meaning since the second term of this expression must be much less than the first term for the approximation to be valid. This means that the following relations must satisfied

$$\alpha^2\left|\frac{k}{2m^3\omega^2} - \frac{16}{9}\frac{k/m^3\omega^2}{1 - \frac{16}{9}\frac{\alpha^2 kT}{m^3\omega^2}}\right| \ll 1, \tag{20a}$$

and

$$\left|1 - \frac{16}{9}\frac{\alpha^2 kT}{m^3\omega^2}\right| > 0. \tag{20b}$$

These two realtions determine the region of the parameters (α, T) where the approximation (16c) is valid.

6. EFFECT ON THE QUANTIZED DISSIPATIVE HARMONIC OSCILLATOR

One may see the effect of dissipation on the quantum harmonic oscillator by observing the change in the eigenvalues of the quantized dissipative system. The quantization can be done by associating an Hermitian operator to the classical dissipative r-Hamiltonian. Using the well-known Weyl's quantization approach (Perelomov, 1986), the Hermitian operator associated to the r-Hamiltonian function (11) for $p > 0$ is given by

$$\widehat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 q^2 + \alpha\left[\omega^2 q^3 - \frac{1}{m^2}(q\hat{p}^2 - i\hbar\hat{p})\right], \tag{21}$$

where the commutation relation $[q, \hat{p}] = i\hbar$ has been used. Defining the non-Hermitian operators

$$a^{+} = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}\, q - \frac{i}{\sqrt{m\omega\hbar}}\,\hat{p}\right] \tag{22a}$$

and

$$a = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}\, q + \frac{i}{\sqrt{m\omega\hbar}}\,\hat{p}\right], \tag{22b}$$

where the commutation relation $[a, a^{+}] = 1$ is satisfied, Eq. (21) can be written as

$$\widehat{H} = \widehat{H}_0 + \widehat{H}_{\mathrm{I}}, \tag{23}$$

where the operators $\widehat{H}_0$ and $\widehat{H}_{\mathrm{I}}$ are defined as

$$\widehat{H}_0 = \hbar\omega\left(a^{+}a + \frac{1}{2}\right) \tag{24a}$$

and

$$\widehat{H}_{\mathrm{I}} = 2\alpha\omega^2\left(\frac{\hbar}{2m\omega}\right)^{3/2}[a^3 + a(a^{+})^2 + a^{+}a^2 + (a^{+})^3 + a - a^{+}]. \tag{24b}$$

It is well known that the eigenfunctions of (24a), $|n\rangle$, have the following properties

$$\begin{aligned} a^{+}a|n\rangle &= n|n\rangle \\ a|n\rangle &= \sqrt{n}\,|n-1\rangle \\ a^{+}|n\rangle &= \sqrt{n+1}\,|n+1\rangle. \end{aligned} \tag{25}$$

There is no contribution of H_{I} to the eigenvalue problem $\widehat{H}\varphi_n = E_n\varphi_n$ at first order in perturbation theory since (24b) contains only odd power terms of the operators a and a^{+}. So, up to second order in perturbation theory (Bohm, 1979), the eigenvalues of (23) are given by

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{3\alpha^2\hbar^2}{m^3}(n^2 + n + 1/2). \tag{26}$$

The second term on the right side of (26) must be much less than the first one for the approximation to be valid. This means that one must have $\alpha \ll \sqrt{m^2\omega/\hbar^2}$.

7. CONCLUSIONS

For the harmonic oscillator with quadratic dissipation, a restricted constant of motion, r-Lagrangian and r-Hamiltonian were deduced for the classical case. The implications of having these function in statistic and quantum mechanics were studied at first order in the dissipation parameter. The results (4), (8), (9a), (11), (12), (19), and (26) have the expected limit for $\alpha \to 0$, and one sees that the contribution of dissipation occurrs at order α^2.

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